ON THE ANALYTIC TORSION OF HYPERBOLIC MANIFOLDS OF FINITE VOLUME

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Abstract. In this paper we study the analytic torsion for a complete oriented hyperbolic manifold of finite volume. This requires the definition of a regularized trace of heat operators. We use the Selberg trace formula to study the asymptotic behavior of the regularized trace for small time. The main result of the paper is a new approach to deal with the weighted orbital integrals on the geometric side of the trace formula.

1. Introduction

Let $G = \mathrm{SO}_0(d, 1)$ and $K = \mathrm{SO}(d)$. Then K is a maximal compact subgroup of G. Let $\widetilde{X} := G/K$. Equipped with a suitably normalized invariant metric, $\widetilde{X}$ is isometric to the hyperbolic space $\mathbb{H}^d$ of dimension d. Let $\Gamma \subset G$ be a lattice, i.e. a discrete subgroup with $\mathrm{vol}(\Gamma\backslash G) < \infty$. Assume that Γ is torsion free. Then $X := \Gamma\backslash\widetilde{X}$ is an oriented hyperbolic d-manifold of finite volume. Let τ be an irreducible finite dimensional complex representation of G. Let E_τ be the flat vector bundle associated to the restriction of τ to Γ. By [MM], E_τ can be equipped with a canonical Hermitian fibre metric, called admissible, which is unique up to scaling. Let $\Delta_p(\tau)$ be the Laplace operator acting in the space of E_τ-valued p-forms with respect to the metrics on X and in E_τ. In [MP2] we introduced the analytic torsion $T_X(\tau)$. If X is compact, $T_X(\tau)$ is defined in the usual way [RS] by

$$\log T_X(\tau) = \frac{1}{2}\sum_{p=1}^{d}(-1)^p p \frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \left(\mathrm{Tr}(e^{-t\Delta_p(\tau)}) - b_p(\tau)\right) t^{s-1}\, dt\right)\bigg|_{s=0}, \tag{1.1}$$

where $b_p(\tau) = \dim\ker(\Delta_p(\tau))$ and the right hand side is defined near $s = 0$ by analytic continuation. In the non-compact case the Laplace operator $\Delta_p(\tau)$ has a nonempty continuous spectrum and hence, $e^{-t\Delta_p(\tau)}$ is not a trace class operator. In [MP2] we introduced the regularized trace $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ of the heat operator which we used to define $T_X(\tau)$ by the analogous formula (1.1) with the usual trace replaced by the regularized trace. In order to show that the Mellin transform of the regularized trace is defined in some half plane and admits a meromorphic extension to the whole complex plane one needs to know the behavior of $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to 0$ and $t \to \infty$. To establish an asymptotic expansion of $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to 0$ we used the Selberg trace formula. The difficult part is to deal

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with the weighted orbital integrals occurring on the geometric side of the trace formula. In fact, in [MP2] we use the invariant trace formula of [Ho1]. In this way the parabolic distributions become invariant distributions. To deal with these distributions we applied the Fourier inversion formula established by W. Hoffmann [Ho2]. This is a very heavy and quite complicated machinery. Moreover at present, except for $\mathrm{SL}(3,\mathbb{R})$, it is not available in higher rank. The main purpose of this paper is to develop a more simplified method to deal with the parabolic contribution to the trace formula, which also has a chance to be extended to the higher rank case.

Next we explain some details of our method. For simplicity we assume that d is odd. We also assume that Γ satisfies (2.13). Let $\nu\colon K \to \mathrm{GL}(V)$ be an irreducible unitary representation of K. Let $\widetilde{E}_\nu \to X$ be the associated homogeneous vector bundle over $\widetilde{X}$ and let $E_\nu = \Gamma\backslash\widetilde{E}_\nu$ be the corresponding locally homogeneous vector bundle over X. Let ∇^ν be the invariant connection in E_ν and let $\Delta_\nu = (\nabla^\nu)^*\nabla^\nu$ be the associated Bochner-Laplace operator. Let A_ν denote the differential operator which is induced in $C^\infty(X, E_\nu)$ by the action of $-\Omega$, where $\Omega \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ is the Casimir element. Then $\Delta_\nu = A_\nu + \nu(\Omega_K)$, where $\Omega_K \in \mathcal{Z}(\mathfrak{k}_\mathbb{C})$ is the Casimir element of K. Note that $\nu(\Omega_K)$ is a scalar. Hence A_ν is essentially self-adjoint and bounded from below. Therefore the heat semigroup e^{-tA_ν} is well defined. The study of the regularized trace of $e^{-t\Delta_p(\tau)}$ can be reduced to the study of the regularized trace of the heat operators e^{-tA_ν}. Let $H^\nu(t,x,y)$ be the kernel of e^{-tA_ν}. For $Y > 1$ sufficiently large let $X(Y)$ be the compact manifold with boundary obtained from X by truncating X at level Y. It was shown in [MP2] that there exists $\alpha(t) \in \mathbb{R}$ such that $\int_{X(Y)} \operatorname{tr} H^\nu(t,x,x)\,dx - \alpha(t)\log Y$ has a limit as $Y \to \infty$. Then we put

$$\operatorname{Tr}\left(e^{-tA_\nu}\right) := \lim_{Y\to\infty}\left(\int_{X(Y)} \operatorname{tr} H^\nu(t,x,x)\;dx - \alpha(t)\log Y\right).$$

Our main result is the following theorem.

Theorem 1.1. *For every $\nu \in \widehat{K}$ there is an asymptotic expansion*

$$\operatorname{Tr}_{\text{reg}}\left(e^{-tA_\nu}\right) \sim \sum_{j=0}^{\infty} a_j(\nu) t^{-d/2+j/2}\log t + \sum_{k=0}^{\infty} b_k(\nu) t^{-d/2+j/2} \tag{1.2}$$

as $t \to 0$. Moreover $a_n(\nu) = 0$.

To prove this theorem, we use the Selberg trace formula as in [MP2]. The connection with the Selberg trace formula is as follows. Let $\widetilde{A}^\nu$ be the lift of A_ν to the universal covering $\widetilde{X}$ of X. The heat operator $e^{-t\widetilde{A}_\nu}$ is a convolution operator with a smooth kernel $H_t^\nu\colon G \to \operatorname{End}(V)$. Let $h_t^\nu \in C^\infty(G)$ be defined by

$$h_t^\nu := \operatorname{tr} H_t^\nu(g), \quad g \in G.$$

In fact, h_t^ν belongs to Harish-Chandra's Schwartz space $\mathcal{C}^1(G)$. In [MP2] it was proved that $\operatorname{Tr}_{\text{reg}}\left(e^{-tA_\nu}\right)$ equals the spectral side of the Selberg trace formula with respect to the test function h_t^ν, where the spectral side means the sum of all terms corresponding to the

discrete and continuous spectrum in the trace formula. Applying the trace formula we are led to the following equality

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-tA_\nu}\right) = I(h_t^\nu) + H(h_t^\nu) + T(h_t^\nu) + T'(h_t^\nu),$$

where I, H, T and T' are certain distributions which are associated to the identity, the hyperbolic conjugacy classes and the parabolic conjugacy classes of Γ, respectively (see [Wa1]). To study the asymptotic behavior of $I(h_t^{\tau,p})$ as $t \to 0$, one can use the Plancherel theorem. Since G has $\mathbb{R}$-rank one, one can use the Fourier inversion formula for regular semi-simple orbital integrals to study $H(h_t^{\tau,p})$. It follows that $H(h_t^{\tau,p})$ is exponentially decreasing as $t \to 0$. The distribution T is invariant and can be expressed in terms of characters. This leads to an asymptotic expansion of $T(h_t^\nu)$ as $t \to 0$. What remains is to deal with the distribution T', which is not invariant. Using standard estimates of heat kernels, it can be reduced to the study of integrals of the form

$$\int_{\mathbb{R}^{d-1}} e^{-f(x)/t} g(x) \log \|x\|\, dx, \tag{1.3}$$

where $g \in C_c^\infty(\mathbb{R}^{d-1})$ and $f \in C^\infty(\mathbb{R}^{d-1})$ has an isolated critical point at $x = 0$ of index zero. Then we use the method of the stationary phase approximation to determine the asymptotic behavior of this integral as $t \to 0$.

The paper is organized as follows. In section 2 we fix notations and collect some basic facts. In section 3 we define the regularized trace of the heat operators e^{-tA_ν} and relate it to the spectral side of the Selberg trace formula where the test function is obtained from the kernel of the heat operator on the universal covering. In section 4 we apply the Selberg trace formula to express the regularized trace through the geometric side of the trace formula. Then we determine the asymptotic behavior of all terms except the weighted orbital integral. Section 5 is a preparatory section where we establish estimates of heat kernels and describe their asymptotic expansion for small time. This is used in section 6 to study the asymptotic behavior of the weighted orbital integrals. Applying the results of the previous section, the problem is reduced to the study of integrals of the form (1.3). To deal with these integrals we apply the method of the stationary phase approximation. This leads finally to the proof of Theorem 1.1. In the last section 7 we discuss the analytic torsion.

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2. Preliminaries

2.1. Let $d = 2n+1$, $n \in \mathbb{N}$. Let either $G = \mathrm{SO}_0(d,1)$, $K = \mathrm{SO}(d)$ or $G = \mathrm{Spin}(d,1)$, $K = \mathrm{Spin}(d)$. Then K is a maximal compact subgroup of G. Put $\widetilde{X} = G/K$. Let

$$G = NAK$$

be the standard Iwasawa decomposition of G and let M be the centralizer of A in K. Then $M = \mathrm{SO}(d-1)$ or $M = \mathrm{Spin}(d-1)$. Let $\mathfrak{g}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{k}$, $\mathfrak{m}$ denote the Lie algebras of G, N, A, K and M, respectively. Define the standard Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(Y) = -Y^t, \quad Y \in \mathfrak{g}.$$

The lift of θ to G will be denoted by the same letter θ. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.1}$$

be the Cartan decomposition of $\mathfrak{g}$ with respect to θ. Let $x_0 = eK \in \tilde{X}$. Then we have a canonical isomorphism

$$T_{x_0}\tilde{X} \cong \mathfrak{p}. \tag{2.2}$$

Define the symmetric bi-linear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ by

$$\langle Y_1, Y_2 \rangle := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}. \tag{2.3}$$

By (2.2) the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{p}$ defines an inner product on $T_{x_0}\tilde{X}$ and therefore an invariant metric on $\tilde{X}$. This metric has constant curvature -1. Then $\tilde{X}$, equipped with this metric, is isometric to the hyperbolic space $\mathbb{H}^d$.

Fix a Cartan subalgebra $\mathfrak{b}$ of $\mathfrak{m}$. Then

$$\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{b}$$

is a Cartan subalgebra of $\mathfrak{g}$. We can identify $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{so}(d+1, \mathbb{C})$. Let $e_1 \in \mathfrak{a}^*$ be the positive restricted root defining $\mathfrak{n}$. Then we fix $e_2, \dots, e_{n+1} \in i\mathfrak{b}^*$ such that the positive roots $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ are chosen as in [Kn2, page 684-685] for the root system D_{n+1}. We let $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$ be the set of roots of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ which do not vanish on $\mathfrak{a}_{\mathbb{C}}$. The positive roots $\Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})$ are chosen such that they are restrictions of elements from $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. For $j = 1, \dots, n+1$ let

$$\rho_j := n + 1 - j. \tag{2.4}$$

Then the half-sums of positive roots ρ_G and ρ_M, respectively, are given by

$$\rho_G := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha = \sum_{j=1}^{n+1} \rho_j e_j; \quad \rho_M := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b}_{\mathbb{C}})} \alpha = \sum_{j=2}^{n+1} \rho_j e_j. \tag{2.5}$$

Let Ω, Ω_K and Ω_M be the Casimir elements of G, K and M, respectively, with respect to the normalized Killing form (2.3).

2.2. Let $\mathbb{Z}\left[\frac{1}{2}\right]^j$ be the set of all $(k_1, \dots, k_j) \in \mathbb{Q}^j$ such that either all k_i are integers or all k_i are half integers. Let $\mathrm{Rep}(G)$ denote the set of finite dimensional irreducible representations τ of G. These are parametrized by their highest weights

$$\Lambda(\tau) = k_1(\tau)e_1 + \cdots + k_{n+1}(\tau)e_{n+1}; \;\; k_1(\tau) \geq k_2(\tau) \geq \cdots \geq k_n(\tau) \geq |k_{n+1}(\tau)|\,, \tag{2.6}$$

where $(k_1(\tau),\dots,k_{n+1}(\tau))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n+1}$ if $G=\operatorname{Spin}(d,1)$ and to $\mathbb{Z}^{n+1}$ if $G=\operatorname{SO}^0(d,1)$. Moreover, the finite dimensional irreducible representations $\nu\in\hat{K}$ of K are parametrized by their highest weights

$$\Lambda(\nu)=k_2(\nu)e_2+\cdots+k_{n+1}(\nu)e_{n+1};\;\; k_2(\nu)\geq k_3(\nu)\geq\cdots\geq k_n(\nu)\geq k_{n+1}(\nu)\geq 0, \tag{2.7}$$

where $(k_2(\nu),\dots,k_{n+1}(\nu))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n}$ if $G=\operatorname{Spin}(d,1)$ and to $\mathbb{Z}^n$ if $G=\operatorname{SO}^0(d,1)$. Finally, the finite dimensional irreducible representations $\sigma\in\hat{M}$ of M are parametrized by their highest weights

$$\Lambda(\sigma)=k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1};\;\; k_2(\sigma)\geq k_3(\sigma)\geq\cdots\geq k_n(\sigma)\geq |k_{n+1}(\sigma)|\,, \tag{2.8}$$

where $(k_2(\sigma),\dots,k_{n+1}(\sigma))$ belongs to $\mathbb{Z}\left[\frac{1}{2}\right]^{n}$, if $G=\operatorname{Spin}(d,1)$, and to $\mathbb{Z}^n$, if $G=\operatorname{SO}^0(d,1)$. For $\nu\in\hat{K}$ and $\sigma\in\hat{M}$ we denote by $[\nu:\sigma]$ the multiplicity of σ in the restriction of ν to M.

Let M' be the normalizer of A in K and let $W(A)=M'/M$ be the restricted Weyl-group. It has order two and it acts on the finite-dimensional representations of M as follows. Let $w_0\in W(A)$ be the non-trivial element and let $m_0\in M'$ be a representative of w_0. Given $\sigma\in\hat{M}$, the representation $w_0\sigma\in\hat{M}$ is defined by

$$w_0\sigma(m)=\sigma(m_0mm_0^{-1}),\quad m\in M.$$

Let $\Lambda(\sigma)=k_2(\sigma)e_2+\cdots+k_{n+1}(\sigma)e_{n+1}$ be the highest weight of σ as in (2.8). Then the highest weight $\Lambda(w_0\sigma)$ of $w_0\sigma$ is given by

$$\Lambda(w_0\sigma)=k_2(\sigma)e_2+\cdots+k_n(\sigma)e_n-k_{n+1}(\sigma)e_{n+1}. \tag{2.9}$$

2.3. Let $P:=NAM$. This is the standard parabolic subgroup of G. We equip $\mathfrak{a}$ with the norm induced from the restriction of the normalized Killing form on $\mathfrak{g}$. Let $H_1\in\mathfrak{a}$ be the unique vector which is of norm one and such that the positive restricted root, implicit in the choice of N, is positive on H_1. Let $\exp:\mathfrak{a}\to A$ be the exponential map. Every $a\in A$ can be written as $a=\exp\log a$, where $\log a\in\mathfrak{a}$ is unique. For $t\in\mathbb{R}$, we let $a(t):=\exp(tH_1)$. If $g\in G$, we define $n(g)\in N$, $H(g)\in\mathbb{R}$ and $\kappa(g)\in K$ by

$$g=n(g)a(H(g))\kappa(g).$$

A given $g\in G$ can always be written in the form

$$g=k_1a(t(g))k_2, \tag{2.10}$$

where $k_1,k_2\in K$ and $t(g)\geq 0$. We note that $t(g)$ is unique and we call it the radial component of g. For $x,y\in\tilde{X}$ let $r(x,y)$ denote the geodesic distance of x and y. Then we have

$$r(g(x_0),x_0)=t(g),\quad g\in G. \tag{2.11}$$

Now let P' be any proper parabolic subgroup of G. Then there exists a $k_{P'}\in K$ such that $P'=N_{P'}A_{P'}M_{P'}$ with $N_{P'}=k_{P'}Nk_{P'}^{-1}$, $A_{P'}=k_{P'}Ak_{P'}^{-1}$, $M_{P'}=k_{P'}Mk_{P'}^{-1}$. We choose

a set of $k_{P'}$'s, which will be fixed from now on. Let $k_P = 1$. We let $a_{P'}(t) := k_{P'}a(t)k_{P'}^{-1}$. If $g \in G$, we define $n_{P'}(g) \in N_{P'}$, $H_{P'}(g) \in \mathbb{R}$ and $\kappa_{P'}(g) \in K$ by

$$g = n_{P'}(g)a_{P'}(H_{P'}(g))\kappa_{P'}(g) \tag{2.12}$$

and we define an identification $\iota_{P'}$ of $(0,\infty)$ with $A_{P'}$ by $\iota_{P'}(t) := a_{P'}(\log(t))$. For $Y > 0$, let $A^0_{P'}[Y] := \iota_{P'}(Y,\infty)$ and $A_{P'}[Y] := \iota_{P'}[Y,\infty)$. For $g \in G$ as in (2.12) we let $y_{P'}(g) := e^{H_{P'}(g)}$.

2.4. Let Γ be a discrete subgroup of G such that $\operatorname{vol}(\Gamma\backslash G) < \infty$. Let $X := \Gamma\backslash\widetilde{X}$. Let $\operatorname{pr}_X : G \to X$ be the projection. A parabolic subgroup P' of G is called a Γ-cuspidal parabolic subgroup if $\Gamma \cap N_{P'}$ is a lattice in $N_{P'}$. We assume that that Γ satisfies the following condition: For every Γ-cuspidal proper parabolic subgroup $P = N_PA_PM_P$ of G we have

$$\Gamma \cap P = \Gamma \cap N_P. \tag{2.13}$$

We note that this condition is satisfied, if Γ is "neat", which means that the group generated by the eigenvalues of any $\gamma \in \Gamma$ contains no roots of unity $\neq 1$.

Let $\mathfrak{P}_\Gamma = \{P_1, \dots, P_{\kappa(\Gamma)}\}$ be a set of representatives of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups of G. The number

$$\kappa(X) := \kappa(\Gamma) = \#\mathfrak{P}_\Gamma \tag{2.14}$$

is finite and equals the number of cusps of X. More precisely, for each $P_i \in \mathfrak{P}_\Gamma$ there exists a $Y_{P_i} > 0$ and there exists a compact connected subset $C = C(Y_{P_1}, \dots, Y_{P_{\kappa(\Gamma)}})$ of G such that in the sense of a disjoint union one has

$$G = \Gamma \cdot C \sqcup \bigsqcup_{i=1}^{\kappa(X)} \Gamma \cdot N_{P_i}A^0_{P_i}[Y_{P_i}]K \tag{2.15}$$

and such that

$$\gamma \cdot N_{P_i}A^0_{P_i}[Y_{P_i}]K \cap N_{P_i}A^0_{P_i}[Y_{P_i}]K \neq \emptyset \Leftrightarrow \gamma \in \Gamma \cap P_i. \tag{2.16}$$

We define the height-function y_{Γ,P_i} on X by

$$y_{\Gamma,P_i}(x) := \sup\{y_{P_i}(g) \colon g \in G,\ \operatorname{pr}_X(g) = x\}. \tag{2.17}$$

By (2.15) and (2.16) the supremum is finite. For $Y \in \mathbb{R}^+$ let

$$X(Y) := \{x \in X \colon y_{\Gamma,P_i}(x) \leq Y,\ i = 1, \dots, \kappa(X)\}. \tag{2.18}$$

2.5. Recall that $d = 2n+1$. For $\sigma \in \hat{M}$ and $\lambda \in \mathbb{R}$ let $\mu_\sigma(\lambda)$ be the Plancherel measure associated to the principal series representation $\pi_{\sigma,\lambda}$. Then, since $rk(G) > rk(K)$, $\mu_\sigma(\lambda)$ is a polynomial in λ of degree $2n$. Let $\langle\cdot,\cdot\rangle$ be the bi-linear form defined by (2.3). Let $\Lambda(\sigma) \in \mathfrak{b}_{\mathbb{C}}^*$ be the highest weight of σ as in (2.8). Then by theorem 13.2 in [Kn1] there exists a constant $c(n)$ such that one has

$$\mu_\sigma(\lambda) = -c(n) \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \frac{\langle i\lambda e_1 + \Lambda(\sigma)+\rho_M,\alpha\rangle}{\langle\rho_G,\alpha\rangle}..$$

The constant $c(n)$ is computed in [Mi2]. By [Mi2], theorem 3.1, one has $c(n) > 0$. For $z \in \mathbb{C}$ let

$$P_\sigma(z) = -c(n) \prod_{\alpha\in\Delta^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h}_{\mathbb{C}})} \frac{\langle ze_1 + \Lambda(\sigma)+\rho_M,\alpha\rangle}{\langle\rho_G,\alpha\rangle}. \tag{2.19}$$

One easily sees that

$$P_\sigma(z) = P_{w_0\sigma}(z). \tag{2.20}$$

3. The regularized trace

Regard G as a principal K-fibre bundle over $\tilde{X}$. By the invariance of $\mathfrak{p}$ under $\mathrm{Ad}(K)$, the assignment

$$T_g^{hor} := \{\frac{d}{dt}\Big|_{t=0} g\exp tX \colon X \in \mathfrak{p}\}$$

defines a horizontal distribution on G. This connection is called the canonical connection. Let ν be a finite-dimensional unitary representation of K on $(V_\nu, \langle\cdot,\cdot\rangle_\nu)$. Let

$$\tilde{E}_\nu := G \times_\nu V_\nu$$

be the associated homogeneous vector bundle over $\tilde{X}$. Then $\langle\cdot,\cdot\rangle_\nu$ induces a G-invariant metric $\tilde{B}_\nu$ on $\tilde{E}_\nu$. Let $\widetilde{\nabla}^\nu$ be the connection on $\tilde{E}_\nu$ induced by the canonical connection. Then $\widetilde{\nabla}^\nu$ is G-invariant. Let

$$E_\nu := \Gamma\backslash(G \times_\nu V_\nu)$$

be the associated locally homogeneous bundle over X. Since $\tilde{B}_\nu$ and $\widetilde{\nabla}^\nu$ are G-invariant, they push down to a metric B_ν and a connection ∇^ν on E_ν. Let

$$C^\infty(G,\nu) := \{f : G \to V_\nu \colon f \in C^\infty,\ f(gk) = \nu(k^{-1})f(g),\ \forall g \in G, \forall k \in K\}. \tag{3.21}$$

Let

$$C^\infty(\Gamma\backslash G,\nu) := \{f \in C^\infty(G,\nu) \colon f(\gamma g) = f(g)\, \forall g \in G, \forall\gamma \in \Gamma\}\,. \tag{3.22}$$

Let $C^\infty(\widetilde{X},\widetilde{E}_\nu)$ (resp. $C^\infty(X,E_\nu)$) denote the space of smooth sections of $\widetilde{E}_\nu$ (resp. E_ν). Then there are canonical isomorphisms

$$\widetilde{\phi} : C^\infty(\widetilde{X},\widetilde{E}_\nu) \cong C^\infty(G,\nu) \quad\text{and}\quad \phi : C^\infty(X,E_\nu) \cong C^\infty(\Gamma\backslash G,\nu) \tag{3.23}$$

(see [Mia, p. 4]). There are also corresponding isometries for the spaces $L^2(\widetilde{X}, \widetilde{E}_\nu)$ and $L^2(X, E_\nu)$ of L^2-sections of $\widetilde{E}_\nu$ and E_ν, respectively. For every $X \in \mathfrak{g}$, $g \in G$, and every $f \in C^\infty(X, E_\nu)$ one has

$$\phi(\nabla^\nu_{L(g)_* X} f)(g) = \frac{d}{dt}\phi(f)(g \exp tX)\big|_{t=0}.$$

Let

$$\Delta_\nu = (\nabla^\nu)^* \nabla^\nu$$

be the Bochner-Laplace operator acting in $C^\infty(X, E_\nu)$. Since X is complete, Δ_ν regarded as linear operator in $L^2(X, E_\nu)$ with domain $C^\infty_c(X, E_\nu)$ is essentially self-adjoint [Che]. By [Mia, Proposition1.1] it follows that on $C^\infty(\Gamma\backslash G, \nu)$ one has

$$\Delta_\nu = -R_\Gamma(\Omega) + \nu(\Omega_K), \tag{3.24}$$

where R_Γ denotes the right regular representation of G on $C^\infty(\Gamma\backslash G, \nu)$. Let A_ν be the differential operator in $C^\infty(X, E_\nu)$ which acts as $-R_\Gamma(\Omega)$ in $C^\infty(\Gamma\backslash G, \nu)$. If ν is irreducible, then $\nu(\Omega_K)$ is a scalar. In general it is an endomorphism of E_ν which commutes with A_ν. It follows from (3.24) that A_ν is a self-adjoint operator which is bounded from below. Therefore, the heat operator e^{-tA_ν} is well defined and we have

$$e^{-t\Delta_\nu} = e^{-t\nu(\Omega_K)} e^{-tA_\nu}. \tag{3.25}$$

Let $H^\nu(t, x, y)$ be the kernel of e^{-tA_ν}. Let $X(Y) \subset X$ be defined by (2.18). For $Y \gg 0$ this is compact manifold with boundary. It follows from [MP2, (5.6)] that there exist smooth functions $a(t)$ and $b(t)$ such that

$$\int_{X(Y)} \operatorname{tr} H^\nu(t, x, x)\, dx = a(t) \log Y + b(t) + o(1)$$

as $Y \to \infty$. Put

$$\operatorname{Tr}_{\text{reg}}\left(e^{-tA_\nu}\right) := b(t). \tag{3.26}$$

In [MP2, (5.6)], the regularized trace $\operatorname{Tr}_{\text{reg}}(e^{-tA_\nu})$ is described explicitly in terms of the discrete spectrum of A_ν and the intertwining operators. To state the formula we need to introduce some notation. Let $\sigma \in \widehat{M}$ with highest weight given by (2.8). Let ρ_j, $j = 1, \dots, n+1$ be defined by (2.4). Put

$$c(\sigma) := \sigma(\Omega_M) - n^2 = \sum_{j=2}^{n+1} (k_j(\sigma) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2, \tag{3.27}$$

where the second equality follows from a standard computation. Let $w \in W(A)$ be the nontrivial element. For $\lambda \in \mathbb{C}$ let

$$\mathbf{C}(\sigma, \lambda)\colon \boldsymbol{\mathcal{E}}(\sigma) \to \boldsymbol{\mathcal{E}}(w\sigma) \tag{3.28}$$

be the intertwining operator defined in [MP2, (3.13)]. Let

$$\widetilde{\boldsymbol{C}}(\sigma, \nu, \lambda)\colon (\boldsymbol{\mathcal{E}}(\sigma) \otimes V_\nu)^K \to (\boldsymbol{\mathcal{E}}(w\sigma) \otimes V_\nu)^K$$

be the restriction of $\boldsymbol{C}(\sigma,\lambda)\otimes \mathrm{Id}_{V_\nu}$ to $(\boldsymbol{\mathcal{E}}(\sigma)\otimes V_\nu)^K$. Furthermore, let A_ν^d denote the restriction of A_ν to the discrete subspace $L^2_d(X,E_\nu)$ of A_ν. Then by [MP2, (5.6)] we have

$$\begin{aligned}\mathrm{Tr}_{\mathrm{reg}}\left(e^{-tA_\nu}\right)=\mathrm{Tr}\left(e^{-tA_\nu^d}\right)+\sum_{\substack{\sigma\in\hat{M};\sigma=w_0\sigma\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)}\frac{\mathrm{Tr}(\widetilde{\boldsymbol{C}}(\sigma,\nu,0))}{4}\\ -\frac{1}{4\pi}\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}} e^{tc(\sigma)}\int_{\mathbb{R}} e^{-t\lambda^2}\,\mathrm{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu,-i\lambda)\frac{d}{dz}\widetilde{\boldsymbol{C}}(\sigma,\nu,i\lambda)\right)d\lambda.\end{aligned}\tag{3.29}$$

It follows from [Wa1, Theorem 8.4] that the right hand side of (3.29) equals the spectral side of the Selberg trace formula applied to e^{-tA_ν}.

4. The trace formula

In this section we apply the Selberg trace formula to study the regularized trace of the heat operator e^{-tA_ν}. To this end we briefly recall the Selberg trace formula. First we introduce the distributions on the geometric side which are associated to the different conjugacy classes of Γ. Let α be a K-finite Schwartz function on G. The contribution of the identity is

$$I(\alpha):=\mathrm{vol}(\Gamma\backslash G)\alpha(1).$$

By [HC2, Theorem 3], the Plancherel theorem can be applied to α. For groups of real rank one which do not possess a compact Cartan subgroup it is stated in [Kn1, Theorem 13.2]. For $\sigma\in\widehat{M}$ and $\lambda\in\mathbb{C}$ let $\pi_{\sigma,\lambda}$ be the principle series representation which we parametrize as in [MP2, Sect. 2.7]. Let $\Theta_{\sigma,\lambda}$ be the character of $\pi_{\sigma,\lambda}$. Let $P_\sigma(z)$ be the polynomial defined by (2.19). Then one has

$$I(\alpha)=\mathrm{vol}(X)\sum_{\substack{\sigma\in\hat{M}\\ [\nu:\sigma]\neq 0}}\int_{\mathbb{R}} P_\sigma(i\lambda)\Theta_{\sigma,\lambda}(\alpha)d\lambda,\tag{4.1}$$

where the sum is finite since α is K-finite. In even dimensions an additional contribution of the discrete series appears.

Next let $\mathrm{C}(\Gamma)_{\mathrm{s}}$ be the set of semi-simple conjugacy classes $[\gamma]$. The contribution of the hyperbolic conjugacy classes is given by

$$H(\alpha):=\int_{\Gamma\backslash G}\sum_{[\gamma]\in\mathrm{C}(\Gamma)_{\mathrm{s}}-[1]}\alpha(x^{-1}\gamma x)dx.$$

By [Wa1, Lemma 8.1] the integral converges absolutely. Its Fourier transform can be computed as follows. Since Γ is assumed to be torsion free, every nontrivial semi-simple element γ is conjugate to an element $m(\gamma)\exp\ell(\gamma)H_1$, $m(\gamma)\in M$. By [Wal, Lemma 6.6], $l(\gamma)>0$ is unique and $m(\gamma)$ is determined up to conjugacy in M. Moreover, $\ell(\gamma)$ is the length of the unique closed geodesic associated to $[\gamma]$. It follows that Γ_γ, the centralizer

of γ in Γ, is infinite cyclic. Let γ_0 denote its generator which is semi-simple too. For $\gamma \in [\Gamma]_S - \{[1]\}$ let $a_\gamma := \exp \ell(\gamma) H_1$ and let

$$L(\gamma,\sigma) := \frac{\overline{\operatorname{Tr}(\sigma)(m_\gamma)}}{\det\left(\operatorname{Id} - \operatorname{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}}\right)} e^{-n\ell(\gamma)}. \tag{4.2}$$

Proceeding as in [Wal] and using [Ga, equation 4.6], one obtains

$$H(\alpha) = \sum_{\sigma\in\hat{M}} \sum_{[\gamma]\in \mathrm{C}(\Gamma)_{\mathrm{s}}-[1]} \frac{l(\gamma_0)}{2\pi} L(\gamma,\sigma) \int_{-\infty}^{\infty} \Theta_{\sigma,\lambda}(\alpha) e^{-il(\gamma)\lambda} d\lambda, \tag{4.3}$$

where the sum is finite since α is K-finite.

Next we describe the parabolic contribution. Put

$$T(\alpha) := \int_K \int_N \alpha(knk^{-1}) dn. \tag{4.4}$$

Let $n \in N$. There exists a unique $Y \in \mathfrak{n}$ such that $n = \exp(Y)$. Put $\|n\| := \|Y\|$. Then let

$$T'(\alpha) := \int_K \int_N \alpha(knk^{-1}) \log\|n\| \; dn \; dk. \tag{4.5}$$

By the Theorem on p. 299 in [OW] there exist constants $C_1(\Gamma)$ and $C_2(\Gamma)$ such that the contribution of the parabolic conjugacy classes equals

$$C_1(\Gamma) T(\alpha) + C_2(\Gamma) T'(\alpha). \tag{4.6}$$

The distributions T and T' are tempered and T is an invariant distribution. Applying the Fourier inversion formula and the Peter-Weyl-Theorem to equation 10.21 in [Kn1], one obtains the Fourier transform of T as:

$$T(\alpha) = \sum_{\sigma\in\hat{M}} \dim(\sigma) \frac{1}{2\pi} \int_{\mathbb{R}} \Theta_{\sigma,\lambda}(\alpha) d\lambda. \tag{4.7}$$

The distribution T'_P is not invariant. One way to deal with this distribution is to make it invariant (see [MP2, (6.15)]) and then apply the Fourier inversion formula of [Ho2]. As explained in the introduction, we will use a different method.

Let $\widetilde{A}_\nu$ be the differential operator in $C^\infty(\widetilde{X}, \widetilde{E}_\nu)$ induced by $-\Omega$. This is the lift of A_ν to $\widetilde{X}$. Let $\widetilde{\Delta}_\nu = (\widetilde{\nabla}^\nu)^* \widetilde{\nabla}^\nu$ be the Bochner-Laplace operator associated to the canonical connection in $\widetilde{E}_\nu$. Then we have

$$\widetilde{\Delta}_\nu = \widetilde{A}_\nu + \nu(\Omega_K). \tag{4.8}$$

Denote by $\widetilde{H}^\nu(t,x,y)$ the kernel of the heat operator $e^{-t\widetilde{A}_\nu}$. Observe that $\widetilde{H}^\nu(t,x,y) \in \operatorname{Hom}((\widetilde{E}_\nu)_y, (\widetilde{E}_\nu)_x)$. For $g \in G$ and $x \in \widetilde{X}$ let $L_g \colon \widetilde{E}_x \to \widetilde{E}_{gx}$ be the isomorphism induced by the left translation. Since $\widetilde{\Delta}_\nu$ commutes with the action of G, the kernel satisfies

$$L_g^{-1} \circ \widetilde{H}^\nu(t,gx,gy) \circ L_g = \widetilde{H}^\nu(t,x,y), \quad x,y \in \widetilde{X},\; g \in G, \tag{4.9}$$

considered as a linear map $\widetilde{E}_y \to \widetilde{E}_x$. Let $x_0 := eK \in \widetilde{X}$. We identify $\widetilde{E}_{x_0}$ with V_ν. Using the isomorphism (3.23), $\widetilde{H}^\nu(t,x,y)$ corresponds to a kernel

$$\widetilde{H}^\nu_t \colon G \times G \to \operatorname{End}(V_\nu),$$

which is defined by

$$\widetilde{H}^\nu_t(g_1,g_2) := L_{g_1}^{-1} \circ \widetilde{H}^\nu(t, g_1x_0, g_2x_0) \circ L_{g_2}. \tag{4.10}$$

By (4.9) it follows that it satisfies

$$\widetilde{H}^\nu_t(gg_1,gg_2) = \widetilde{H}^\nu_t(g_1,g_2), \quad g, g_1, g_2 \in G, \tag{4.11}$$

and

$$\widetilde{H}^\nu_t(g_1k_1,g_2k_2) = \nu(k_1)^{-1} \circ \widetilde{H}^\nu_t(g_1,g_2) \circ \nu(k_2), \quad k_1,k_2 \in K,\ g \in G. \tag{4.12}$$

Using (4.11), we can identify $\widetilde{H}^\nu_t$ with a map

$$H^\nu_t \colon G \to \operatorname{End}(V_\nu)$$

by

$$H^\nu_t(g) := \widetilde{H}^\nu_t(e,g), \quad g \in G. \tag{4.13}$$

Then H^ν_t belongs to $(\mathcal{C}^1(G) \otimes \operatorname{End}(V_\nu))^K$ and satisfies

$$H^\nu_t(k_1gk_2) = \nu(k_1) \circ H^\nu_t(g) \circ \nu(k_2), \quad k_1,k_2 \in K,\ g \in G. \tag{4.14}$$

Let h^ν_t be defined by

$$h^\nu_t(g) := \operatorname{tr} H^\nu_t(g), \quad g \in G. \tag{4.15}$$

Then belongs h^ν_t to $\mathcal{C}^1(G)$ (see [BM]). If we apply the Selberg trace formula [Wa1, Theorem 8.4] to (3.29) and use (4.6), we obtain the following theorem.

Theorem 4.1. *For all $t > 0$ we have*

$$\operatorname{Tr}_{reg}\left(e^{-tA_\nu}\right) = I(h^\nu_t) + H(h^\nu_t) + C_1(\Gamma)T(h^\nu_t) + C_2(\Gamma)T'(h^\nu_t).$$

This theorem can be used to determine the asymptotic behavior of $\operatorname{Tr}_{reg}\left(e^{-tA_\nu}\right)$ as $t \to 0$. For $I(h^\nu_t)$ we use (4.1). The character $\Theta_{\sigma,\lambda}(h^\nu_t)$ is computed by [MP2, Proposition 4.1]. We have

$$\Theta_{\sigma,\lambda}(h^\nu_t) = e^{t(c(\sigma)-\lambda^2)}. \tag{4.16}$$

By (4.1) it follows that

$$I(h^\nu_t) = \operatorname{vol}(X) \sum_{\substack{\sigma \in \hat{M} \\ [\nu:\sigma] \neq 0}} e^{tc(\sigma)} \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda)\, d\lambda.$$

Now recall that $P_\sigma(z)$ is an even polynomial of degree $2n = d-1$. Hence we obtain an expansion

$$I(h_t^\nu) = t^{-d/2}\sum_{j=0}^{\infty} a_j t^j. \tag{4.17}$$

Using (4.16) and (4.3), it follows that

$$H(h_t^\nu) = O(e^{-c/t}), \quad 0 < t \le 1. \tag{4.18}$$

Furthermore, by (4.7) and (4.16) we get

$$T(h_t^\nu) = t^{-1/2}\sum_{j=0}^{\infty} b_j t^j. \tag{4.19}$$

It remains to determine the asymptotic behavior of $T'(h_t^\nu)$. This will be done in the next sections.

5. Heat kernel estimates

In this section we study the kernel $K^\nu(t,x,y)$ of $e^{-t\Delta_\nu}$. Observe that $K^\nu(t,x,y) \in \operatorname{Hom}((\widetilde{E}_\nu)_y, (\widetilde{E}_\nu)_x)$. Denote by $|K^\nu(t,x,y)|$ the norm of this homomorphism. Furthermore, let $r(x,y)$ denote the geodesic distance of $x, y \in \widetilde{X}$. We have

Proposition 5.1. *For every $T > 0$ there exists $C > 0$ such that for all $\nu \in \widehat{K}$ we have*

$$|K^\nu(t,x,y)| \le Ct^{-d/2} exp\left(-\frac{r^2(x,y)}{4t}\right)$$

for all $0 < t \le T$ and $x, y \in \widetilde{X}$, where $d = \dim \widetilde{X}$.

Proof. If ν is irreducible, this is proved in [Mu1, Proposition 3.2]. However, the proof does not make any use of the irreducibility of ν. So it extends without any change to the case of finite-dimensional representations. □

By (4.8) the kernel $\widetilde{H}^\nu(t,x,y)$ of e^{-tA_ν} is closely related to the kernel $K^\nu(t,x,y)$. If ν is irreducible, $\nu(\Omega_K)$ is a scalar and we have

$$\widetilde{H}^\nu(t,x,y) = e^{t\nu(\Omega_K)}K^\nu(t,x,y). \tag{5.1}$$

Let $h_t^\nu \in \mathcal{C}^1(G)$ be defined by (4.15). Note that for each $g \in G$, $L_g\colon E_x \to E_{gx}$ is an isometry. Thus using (4.10), the definition of H_t^ν, (5.1), and Proposition 5.1, we get

Corollary 5.2. *Let $d = \dim \widetilde{X}$. For all $T > 0$ there exists $C > 0$ such that we have*

$$|h_t^\nu(g)| \le Ct^{-d/2}\exp\left(-\frac{r^2(gx_0,x_0)}{4t}\right)$$

for all $0 < t \le T$ and $g \in G$.

Next we turn to the asymptotic expansion of the heat kernel. Let $d_{\mathbf{x}}\exp_{x_0}$ be the differential of the exponential map $\exp_{x_0}\colon T_{x_0}\widetilde{X}\to\widetilde{X}$ at the point $\mathbf{x}\in T_{x_0}\widetilde{X}$. It is a map from $T_{x_0}\widetilde{X}$ to $T_x\widetilde{X}$, where $x=\exp_{x_0}(\mathbf{x})$. Let

$$j(\mathbf{x}):=|\det(d_{\mathbf{x}}\exp_{x_0})| \tag{5.2}$$

be the Jacobian, taken with respect to the inner products in the tangent spaces. Note that

$$j(\mathbf{x})=|\det(g_{ij}(\mathbf{x}))|^{1/2}. \tag{5.3}$$

Write $y=\exp_x(\mathbf{y})$, with $\mathbf{y}\in T_x\tilde{X}$. Let $\varepsilon>0$ be sufficiently small. Let $\psi\in C^\infty(\mathbb{R})$ with $\psi(u)=1$ for $u<\varepsilon$ and $\psi(u)=0$ for $u>2\varepsilon$.

Proposition 5.3. *Let $d=\dim\widetilde{X}$. Let (ν,V_ν) be a finite-dimensional unitary representation of K. There exist smooth sections $\Phi_i^\nu\in C^\infty(\widetilde{X}\times\tilde{X},\widetilde{E}_\nu\boxtimes\widetilde{E}_\nu^*)$, $i\in\mathbb{N}_0$, such that for every $N\in\mathbb{N}$*

$$K^\nu(t,x,y)=t^{-d/2}\psi(d(x,y))\exp\left(-\frac{r^2(x,y)}{4t}\right)\sum_{i=0}^N\Phi_i^\nu(x,y)j(x,y)^{-1/2}t^i \\ +O(t^{N+1-d/2}), \tag{5.4}$$

uniformly for $0<t\le 1$. Moreover the leading term $\Phi_0^\nu(x,y)$ is equal to the parallel transport $\tau(x,y)\colon(\widetilde{E}_\nu)_y\to(\widetilde{E}_\nu)_x$ with respect to the connection ∇^ν along the unique geodesic joining x and y.

Proof. Let $\Gamma\subset G$ be a co-compact torsion free lattice. It exists by [Bo]. Let $X=\Gamma\backslash\widetilde{X}$ and $E_\nu=\Gamma\backslash\widetilde{E}_\nu$. As in [Do, Sect 3], the proof can be reduced to the compact case, which follows from [BGV, Theorem 2.30]. □

Let $\mathfrak{p}$ be as in (2.1). We recall that the mapping

$$\varphi\colon\mathfrak{p}\times K\to G,$$

defined by $\varphi(Y,k)=\exp(Y)\cdot k$ is a diffeomorphism [He, Ch. VI, Theorem 1.1]. Thus each $g\in G$ can be uniquely written as

$$g=\exp(Y(g))\cdot k(g),\quad Y(g)\in\mathfrak{p},\ k(g)\in K. \tag{5.5}$$

Using Proposition 5.3 and (5.1), we obtain the following corollary.

Corollary 5.4. *There exist $a_i^\nu\in C^\infty(G)$ such that*

$$h_t^\nu(g)=t^{-d/2}\psi(d(gx_0,x_0))\exp\left(-\frac{r^2(gx_0,x_0)}{4t}\right)\sum_{i=0}^N a_i^\nu(g)t^i+O(t^{N+1-d/2}) \tag{5.6}$$

which holds for $0<t\le 1$. Moreover the leading coefficient a_0^ν is given by

$$a_0^\nu(g)=\operatorname{tr}(\nu(k(g)))\cdot j(x_0,gx_0)^{-1/2}. \tag{5.7}$$

Proof. By (4.10) and (4.13) we have

$$H_t^\nu(g) = H^\nu(t, x_0, gx_0) \circ L_g, \quad g \in G.$$

Put

$$a_i^\nu(g) := \operatorname{tr}(\Phi_i^\nu(x_0, gx_0) \circ L_g) \cdot j(x_0, gx_0)^{-1/2}, \quad g \in G. \tag{5.8}$$

Then (5.6) follows immediately from (5.4) and the definition of h_t^ν. To prove the second statement, we recall that $\Phi_0^\nu(x, y)$ is the parallel transport $\tau(x, y)$ with respect to the canonical connection of $\widetilde{E}_\nu$ along the geodesic connecting x and y. Let $g = \exp(Y) \cdot k$, $Y \in \mathfrak{p}$, $k \in K$. Then the geodesic connecting x_0 and gx_0 is the curve $\gamma(t) = \exp(tY)x_0$, $t \in [0, 1]$ (see [He, Ch. IV, Theorem 3.3]). The parallel transport along $\gamma(t)$ equals $L_{\exp(Y)}$. Thus $\Phi_0^\nu(x_0, gx_0) = L_{\exp(Y)}^{-1}$. Hence we get.

$$\Phi_0^\nu(x_0, gx_0) \circ L_g = L_k = \nu(k).$$

Together with (5.8) the claim follows. □

6. Weighted orbital integrals

The weighted orbital integral is given by (4.5). We apply this to h_t^ν. By (4.14) it follows that h_t^ν is invariant under conjugation by $k \in K$. Thus we get

$$T'_P(h_t^\nu) = \int_N h_t^\nu(n) \log \|n\| \, dn. \tag{6.1}$$

We fix an isometric identification of $\mathbb{R}^{d-1}$ with $\mathfrak{n}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\theta$ defined by

$$\langle Y_1, Y_2 \rangle_\theta := \langle Y_1, \theta(Y_2) \rangle, \quad Y_1, Y_2 \in \mathfrak{g}.$$

Explicitly it is given by

$$x \in \mathbb{R}^{d-1} \mapsto Y(x) := \begin{pmatrix} 0 & -x & x \\ x^T & 0 & 0 \\ x^T & 0 & 0 \end{pmatrix}, \tag{6.2}$$

where we consider x as a column. Furthermore we identify $\mathfrak{n}$ with N via the exponential map. Put

$$n(x) := \exp(Y(x)) \in N, \quad x \in \mathbb{R}^{d-1}.$$

We note that

$$n(x) = \operatorname{Id} + Y(x) + \frac{1}{2} Y(x)^2. \tag{6.3}$$

Then we get

$$T'(h_t^\nu) = \int_{\mathbb{R}^{d-1}} h_t^\nu\,(n(x)) \log \|x\| \, dx. \tag{6.4}$$

For $\varepsilon > 0$ let $B(\varepsilon) \subset \mathbb{R}^{d-1}$ denote the ball of radius ε centered at 0 and let

$$U(\varepsilon) := \mathbb{R}^{d-1} - B(\varepsilon).$$

We decompose the integral as

$$\int_{\mathbb{R}^{d-1}} = \int_{B(\varepsilon)} + \int_{U(\varepsilon)}.$$

Put

$$r(x) := r(n(x)x_0, x_0), \quad x \in \mathbb{R}^{d-1}. \tag{6.5}$$

We need some properties of this function.

Lemma 6.1. *We have*

$$r(x) = \operatorname{arcosh}\left(1 + \frac{\|x\|^2}{2}\right), \quad x \in \mathbb{R}^{d-1}. \tag{6.6}$$

Proof. Let $t(x) \geq 0$ be the radial component of $n(x)$, defined by (2.10). By [Wa1, Lemma 7.1] we have

$$t(x) = \operatorname{arcosh}\left(1 + \frac{\|x\|^2}{2}\right).$$

The lemma follows from (2.11). □

Now note that

$$\operatorname{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right), \quad x \geq 1.$$

Thus

$$r(x) \geq \ln\left(1 + \frac{\|x\|^2}{2}\right), \quad x \in \mathbb{R}^{d-1}. \tag{6.7}$$

This implies that for all $t > 0$ we have

$$\int_{\mathbb{R}^{d-1}} \exp\left(-\frac{r^2(x)}{4t}\right) |\log \|x\|| \, dx < \infty. \tag{6.8}$$

Using $\operatorname{arcosh}'(x) = (x^2 - 1)^{-1/2}$, we get

$$\frac{d}{dr} \operatorname{arcosh}\left(1 + \frac{r^2}{2}\right) = \frac{1}{\sqrt{1 + r^2/4}}, \quad r \geq 0.$$

Thus

$$r(x_1) > r(x_2), \quad \text{if } \|x_1\| > \|x_2\|. \tag{6.9}$$

Now observe that $\operatorname{arcosh}(1 + x^2/2)$ has a Taylor series expansion of the form

$$\operatorname{arcosh}\left(1 + \frac{x^2}{2}\right) = x + \sum_{k=1}^{\infty} a_k x^{2k+1}$$

which converges for $|x| < 1/2$. This follows from [GR, 1.631.2] together [GR, 1.641.2]. Thus $r^2(x)$ is C^∞ and we have

$$r^2(x) = \|x\|^2 + R(x) \tag{6.10}$$

with

$$|R(x)| \leq C\|x\|^4, \quad \|x\| \leq 1/4.$$

Thus $r^2(x)$ has a non-degenerate critical point at $x = 0$ of index $(d-1, 0)$.

Now we turn to the estimation of the orbital integral. Put

$$c(\varepsilon) := \operatorname{arcosh}(1 + \varepsilon^2/2).$$

By Corollary 5.2, (6.8) and (6.9) we get

$$\begin{aligned}\left|\int_{U(\varepsilon)} h_t^\nu(n(x)) \log \|x\|\, dx\right| &\leq C t^{-d/2} \int_{U(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) |\log \|x\||dx \\ &\leq C_1 t^{-d/2} \exp\left(-\frac{c(\varepsilon)}{8t}\right)\end{aligned} \tag{6.11}$$

for $0 < t \leq 1$. To deal with the integral over $B(\varepsilon)$, we use (5.6). This gives

$$\begin{aligned}\int_{B(\varepsilon)} h_t^\nu(n(x)) \log \|x\|\, dx =& t^{-d/2} \sum_{i=1}^N t^i \int_{B(\varepsilon)} \exp\left(-\frac{r^2(x)}{4t}\right) \psi(x) a_i^\nu(x) \log \| x \|\, dx \\ &+ O(t^{N+1-d/2})\end{aligned} \tag{6.12}$$

for $0 < t \leq 1$, where $\psi \in C_c^\infty(B(\varepsilon))$. Put $m = d-1$. The integrals on the right hand side are of the form

$$I(\lambda) = \int_{\mathbb{R}^m} e^{-\lambda f(x)} g(x) \log \| x \|\, dx, \tag{6.13}$$

where $\lambda > 0$, $g \in C_c^\infty(\mathbb{R}^m)$, $\operatorname{supp} g \subset B(\varepsilon)$, and f satisfies

$$f(x) = \|x\|^2 + R(x), \quad |R(x)| < C\|x\|^4, \;\; \|x\| < \varepsilon, \tag{6.14}$$

and f has no critical points in $\operatorname{supp} g \setminus \{0\}$. Our goal is to derive an asymptotic expansion for $I(\lambda)$ as $\lambda \to \infty$. To begin with, we first show that f can be replaced by $\|x\|^2$. We proceed as in Hörmander's proof of the stationary phase approximation [Hor, Theorem 7.7.5]. On $B(\varepsilon) \setminus \{0\}$ we consider the following differential operator

$$L := -\frac{1}{\|f'(x)\|^2} \sum_{j=1}^m \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}. \tag{6.15}$$

Note that the formally adjoint operator L^* is given by

$$L^* = \sum_{j=1}^m \frac{\partial}{\partial x_j} \frac{1}{\|f'(x)\|^2} \frac{\partial f}{\partial x_j}, \tag{6.16}$$

where the factors act as multiplication operators. Since integration by parts arguments will introduce singularities, we make some more general assumptions. Suppose that $g \in$

$C^1(\mathbb{R}^m \setminus \{0\})$ with support in $B(\varepsilon)$ and with $|D^\alpha g| \le C\|x\|^{3-|\alpha|}$ for $|\alpha| \le 1$. Using the divergence theorem in the last step, we have

$$
\begin{aligned}
I(\lambda) &= \lim_{r\to 0} \int_{\|x\|\ge r} e^{-\lambda f(x)} g(x) \log\|x\|\, dx = \lambda^{-1} \lim_{r\to 0} \int_{\|x\|\ge r} (Le^{-\lambda f(\cdot)})(x) g(x) \log\|x\|\, dx \\
&= \lambda^{-1} \lim_{r\to 0} \int_{\|x\|\ge r} e^{-\lambda f(x)} L^*(g \log\|\cdot\|)(x)\, dx \\
&\quad + \lambda^{-1} \lim_{r\to 0} \int_{\|x\|= r} \|f'(x)\|^{-2} \langle \nu, \nabla f\rangle(x) e^{-\lambda f(x)} g(x) \log\|x\|\, dS(x),
\end{aligned}
$$

where ν is the Durward unit normal vector field to $\partial(\mathbb{R}^m \setminus B(r))$. By the assumption on f, there exists $C > 0$ such that

$$
\|f'(x)\|^{-2} \le C\|x\|^{-2}, \quad \|\nabla f(x)\| \le C\|x\|.
$$

Together with the assumptions on g, it follows that the integrand in the surface integral is bounded on $B(\varepsilon)$. Thus the surface integral has limit 0 as $r \to 0$. Furthermore, by (6.16) we have

$$
L^*(g \log\|\cdot\|)(x) = \sum_{j=1}^m \frac{\partial}{\partial x_j} \left(\frac{1}{\|f'(x)\|^2} \frac{\partial f}{\partial x_j}(x) g(x) \log\|x\| \right).
$$

Using the assumptions on f and g, it follows that $L^*(g \log\|\cdot\|)$ is bounded, and therefore

$$
I(\lambda) = \lambda^{-1} \lim_{r\to 0} \int_{\|x\|\ge r} e^{-\lambda f(x)} L^*(g \log\|\cdot\|)(x)\, dx = \lambda^{-1} \int_{\mathbb{R}^m} e^{-\lambda f(x)} L^*(g \log\|\cdot\|)(x)\, dx.
$$

Using the properties of f and g, we get

$$
|I(\lambda)| \le C(f)\lambda^{-1} \sum_{|\alpha|\le 1} \sup_{x\in\mathbb{R}^m} |D^\alpha(g(x)\log\|x\|) \cdot \|x\|^{-2+|\alpha|}|.
$$

Now let $k \in \mathbb{N}$ and assume that $g \in C^k(\mathbb{R}^m \setminus \{0\})$ has support in $B(\varepsilon)$ and satisfies

$$
|D^\alpha g(x)| \le C\|x\|^{2k+1-|\alpha|}, \quad \text{for} \quad |\alpha| \le k.
$$

Let $u(x) = g(x)\log\|x\|$, $x \in \mathbb{R}^m \setminus \{0\}$. Then $u \in C^k(\mathbb{R}^m \setminus \{0\})$, with support in $B(\varepsilon)$ and with $|D^\alpha u(x)| \le C\|x\|^{2k-|\alpha|}$ for $|\alpha| \le k$. Then it follows that $L^*(u) \in C^{k-1}(\mathbb{R}^m \setminus \{0\})$, with support in $B(\varepsilon)$ and with $|D^\alpha L^*(u)(x)| \le C\|x\|^{2(k-1)-|\alpha|}$ for $|\alpha| \le k$. Thus we can proceed by induction to conclude that

$$
|I(\lambda)| \le C(f)\lambda^{-k} \sum_{|\alpha|\le k} \sup_{x\in\mathbb{R}^m} \left| D^\alpha(g(x)\log\|x\|) \cdot \|x\|^{-2k+|\alpha|} \right|. \tag{6.17}
$$

We introduce the following auxiliary functions

$$
f_s(x) := \|x\|^2 + sR(x), \quad s \in [0,1]. \tag{6.18}
$$

Then $f_1 = f$. Let

$$
I(\lambda, s) := \int_{\mathbb{R}^m} e^{-\lambda f_s(x)} g(x) \log\|x\|\, dx.
$$

Differentiating in s 2k times yields

$$I^{(2k)}(\lambda,s)=\lambda^{2k}\int_{\mathbb{R}^m} e^{-\lambda f_s(x)}R(x)^{2k}g(x)\log\|x\|\,dx. \tag{6.19}$$

Now let $g\in C^\infty(\mathbb{R}^m)$ with $\operatorname{supp} g\subset B(\varepsilon)$. Let $u(x)=R(x)^{2k}g(x)\log\|x\|$. Then $u\in C^\infty(\mathbb{R}^m\setminus\{0\})$ with support in $B(\varepsilon)$ and by (6.14) it follows that $|D^\alpha u(x)|\le C\|x\|^{6k-|\alpha|}$, $\|x\|<\varepsilon$. Applying (6.17) with $3k$ in place of k, and (6.19), we get

$$|I^{2k}(\lambda,s)|\le C\lambda^{-k}.$$

By Taylor's theorem, we have

$$\left|I(\lambda,1)-\sum_{j<2k}\frac{1}{j!}I^{(j)}(\lambda,0)\right|\le\sup_{s\in[0,1]}\frac{1}{(2k)!}\left|I^{2k}(\lambda,s)\right|. \tag{6.20}$$

Thus modulo λ^{-k} decay, it suffices to study the asymptotic expansion of $I^{(j)}(\lambda,0)$. Each of these integrals is of the form

$$I_0(\lambda)=\int_{\mathbb{R}^m} e^{-\lambda\|x\|^2}g(x)\log\|x\|\,dx, \tag{6.21}$$

where $g\in C_c^\infty(\mathbb{R}^m)$ with support in $B(\varepsilon)$. Let $N\in\mathbb{N}$. By Taylor's theorem we have

$$\left|g(x)-\sum_{|\alpha|\le N}\frac{(D^\alpha g)(0)}{\alpha!}x^\alpha\right|\le\left(\sum_{|\alpha|=N+1}\frac{1}{\alpha!}\sup_{x\in\mathbb{R}^m}|(D^\alpha g)(x)|\right)\cdot\|x\|^{N+1}$$

for $x\in B(\varepsilon)$. Now for $\lambda\ge 1$ we have

$$\begin{aligned}\left|\int_{B(\varepsilon)} e^{-\lambda\|x\|^2}\|x\|^{N+1}\log\|x\|\,dx\right|&\le\int_{\mathbb{R}^m} e^{-\lambda\|x\|^2}\|x\|^{N+1}\left|\log\|x\|\right|\,dx\\&=O(\lambda^{-(m+N+1)/2}(1+\log\lambda)).\end{aligned}$$

Furthermore, for $\lambda\ge 1$ we have

$$\int_{B(\varepsilon)} e^{-\lambda\|x\|^2}x^\alpha\log\|x\|\,dx=\int_{\mathbb{R}^m} e^{-\lambda\|x\|^2}x^\alpha\log\|x\|\,dx+O(e^{-\varepsilon\lambda/2}).$$

Changing variables $x\mapsto x/\sqrt{\lambda}$ in the integral on the right hand side, we get

$$\int_{\mathbb{R}^m} e^{-\lambda\|x\|^2}x^\alpha\log\|x\|\,dx=c_1\lambda^{-m/2-|\alpha|/2}+c_2\lambda^{-m/2-|\alpha|/2}\log\lambda$$

for some constants c_1 and c_2. Summarizing, it follows that for every $N\in\mathbb{N}$ we have an expansion

$$I(\lambda)=\sum_{j=0}^N a_j\lambda^{-m/2-j/2}\log\lambda+\sum_{j=0}^N b_j\lambda^{-m/2-j/2}+O(\lambda^{-(m+N)/2}) \tag{6.22}$$

as $\lambda\to\infty$. Now the integrals on the right hand side of (6.12) are of the form $I(1/(4t))$. Combining (6.4), (6.11), (6.12), and (6.22), and using that $m=d-1$, we get

Proposition 6.2. *For every $N \in \mathbb{N}$ we have*

$$T'_P(h_t^\nu) = \sum_{j=0}^{N} c_j(\nu) t^{(j-1)/2} \log t + \sum_{j=0}^{N} d_j(\nu) t^{(j-1)/2} + O(t^{(N+1)/2})$$

as $t \to 0$. Moreover $c_1(\nu) = 0$.

Proof. Only the last statement needs to be proved. The only terms that can make a contribution to $c_1(\nu)$ are the Taylor coefficients of $a_0^\nu(n(x))$ of degree 1, i.e.,

$$c_1(\nu) = \sum_{i=1}^{d-1} \frac{\partial}{\partial x_i} a_0^\nu(n(x))\big|_{x=0} \int_{\mathbb{R}^{d-1}} e^{-\|x\|^2} x_i \, dx.$$

By (5.7) we have

$$a_0^\nu(n(x)) = \operatorname{tr}(k(n(x))) \cdot j(x_0, n(x)x_0)^{-1/2}.$$

We have

$$j(x_0, n(x)x_0) = \frac{\sinh(r(x))}{r(x)},$$

where $r(x)$ is given by (6.5). Using (6.10), if follows that

$$\frac{\partial}{\partial x_i} j(x_0, n(x)x_0)^{-1/2}\big|_{x=0} = 0, \quad i = 1, \dots, d. \tag{6.23}$$

□

Combining Proposition 6.2 with (4.17), (4.18) and (4.19), Theorem 1.1 follows.

7. Analytic torsion

Let τ be an irreducible finite dimensional representation of G on V_τ. Let E'_τ be the flat vector bundle associated to the restriction of τ to Γ. Then E'_τ is canonically isomorphic to the locally homogeneous vector bundle E_τ associated to $\tau|_K$. By [MM], there exists an inner product $\langle \cdot, \cdot \rangle$ on V_τ such that

(1) $\langle \tau(Y)u, v \rangle = -\langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{k}$, $u, v \in V_\tau$
(2) $\langle \tau(Y)u, v \rangle = \langle u, \tau(Y)v \rangle$ for all $Y \in \mathfrak{p}$, $u, v \in V_\tau$.

Such an inner product is called admissible. It is unique up to scaling. Fix an admissible inner product. Since $\tau|_K$ is unitary with respect to this inner product, it induces a metric on E_τ which will be called admissible too. Note that for each p, the vector bundle $\Lambda^p(E_\tau) = \Lambda^p T^*X \otimes E_\tau$ is associated with the representation

$$\nu_p(\tau) := \Lambda^p \operatorname{Ad}^* \otimes \tau : \; K \to \operatorname{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau), \tag{7.1}$$

i.e., there is a canonical isomorphism

$$\Lambda^p(E_\tau) \cong \Gamma \backslash (G \times_{\nu_p(\tau)} (\Lambda^p \mathfrak{p}^* \otimes V_\tau)). \tag{7.2}$$

By (3.23) there is an isomorphism

$$\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma\backslash G, \nu_p(\tau)), \tag{7.3}$$

and a corresponding isomorphism for the L^2-spaces. Let $\Delta_p(\tau)$ be the Hodge-Laplacian on $\Lambda^p(X, E_\tau)$ with respect to the admissible inner product. By (6.9) in [MM] it follows that with respect to (7.3) one has

$$\Delta_p(\tau) = -R_\Gamma(\Omega) + \tau(\Omega)\,\mathrm{Id}, \tag{7.4}$$

where R_Γ is the right regular representation. Let $A_{\nu_p(\tau)}$ be the differential operator induced by $-R_\Gamma(\Omega)$ in $C^\infty(X, E_\nu)$. We note that $\tau(\Omega)$ is a scalar which can be computed as follows. If $\Lambda(\tau) = k_1(\tau)e_1 + \dots k_{n+1}(\tau)e_{n+1}$ is the highest weight of τ, then

$$\tau(\Omega) = \sum_{j=1}^{n+1} (k_j(\tau) + \rho_j)^2 - \sum_{j=1}^{n+1} \rho_j^2. \tag{7.5}$$

For $G = \mathrm{Spin}(2n+1, 1)$ this was proved in [MP1, sect. 2]. For $G = \mathrm{SO}_0(2n+1, 1)$, the proof is similar. Thus we get

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) = e^{-t\tau(\Omega)}\,\mathrm{Tr}_{\mathrm{reg}}\left(e^{-tA_{\nu_p(\tau)}}\right). \tag{7.6}$$

In order to define the analytic torsion, we have to determine the asymptotic behavior of $\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to 0$ and $t \to \infty$. By Theorem 1.1 it follows that there is an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) \sim \sum_{j=0}^{\infty} a_j(p,\tau) t^{-d/2+j/2} \log t + \sum_{k=0}^{\infty} b_k(p,\tau) t^{-d/2+j/2} \tag{7.7}$$

as $t \to 0$. Moreover $a_n(p, \tau) = 0$. To determine the asymptotic behavior of the regularizes trace as $t \to \infty$, we use (3.29). Let $0 \le \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $\Delta_p(\tau)$. By (7.4) and (3.29) we have

$$\begin{aligned} \mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) &= \sum_j e^{-t\lambda_j} + \sum_{\substack{\sigma\in\hat M;\sigma=w_0\sigma \\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))} \frac{\mathrm{Tr}(\widetilde{\boldsymbol{C}}(\sigma,\nu_p(\tau),0))}{4} \\ &\quad - \frac{1}{4\pi} \sum_{\substack{\sigma\in\hat M \\ [\nu_p(\tau):\sigma]\neq 0}} e^{-t(\tau(\Omega)-c(\sigma))} \\ &\quad \cdot \int_{\mathbb{R}} e^{-t\lambda^2}\,\mathrm{Tr}\left(\widetilde{\boldsymbol{C}}(\sigma,\nu_p(\tau),-i\lambda)\frac{d}{dz}\widetilde{\boldsymbol{C}}(\sigma,\nu_p(\tau),i\lambda)\right) d\lambda. \end{aligned} \tag{7.8}$$

Assume that $d = 2n+1$. Let $h_p(\tau) := \dim(\ker \Delta_p(\tau) \cap L^2)$. Using (7.8) and [MP2, Lemmas 7.1, 7.2], it follows that there is an asymptotic expansion

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) \sim h_p(\tau) + \sum_{j=1}^{\infty} c_j t^{-j/2}, \quad t \to \infty \tag{7.9}$$

Combined with (7.7) we can define the spectral zeta function by

$$\begin{aligned}\zeta_p(s;\tau) := \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)-h_p(\tau)\right)\,dt\\ +\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)-h_p(\tau)\right)\,dt.\end{aligned} \tag{7.10}$$

By (7.7) the first integral on the right converges in the half-plane $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. By (7.9), the second integral converges in the half-plane $\operatorname{Re}(s) < 1/2$ and also admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$.

Now assume that $\tau \ncong \tau_\theta$. Let the highest weight $\Lambda(\tau)$ be given by (2.6). The highest weight $\Lambda(\tau_\theta)$ of τ_θ is given by

$$\Lambda(\tau_\theta) = k_1(\tau)e_1 + \cdots + k_n(\tau)e_n - k_{n+1}(\tau)e_{n+1}.$$

Therefore, the condition $\tau \ncong \tau_\theta$ implies $k_{n+1}(\tau) \neq 0$. Then by [MP2, Lemma 7.1] we have $\tau(\Omega) - c(\sigma) > 0$ for all $\sigma \in \hat{M}$ with $[\nu_p(\tau) \colon \sigma] \neq 0$ and $p = 0, \dots, d$. Furthermore by [MP2, Lemma 7.3] we have $\ker(\Delta_p(\tau) \cap L^2) = 0$, $p = 0, \dots, d$. By (7.8) it follows that there exist $C, c > 0$ such that for all $p = 0, \dots, d$:

$$\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) \le Ce^{-ct}, \quad t \ge 1. \tag{7.11}$$

Using (6.2), it follows that $\zeta_p(s;\tau)$ can be defined as in the compact case by

$$\zeta_p(s;\tau) := \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right)\,dt. \tag{7.12}$$

The integral converges absolutely and uniformly on compact subsets of $\operatorname{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. We define the regularized determinant of $\Delta_p(\tau)$ as in the compact case by

$$\det \Delta_p(\tau) := \exp\left(-\frac{d}{ds}\zeta_p(s;\tau)\big|_{s=0}\right). \tag{7.13}$$

In analogy to the compact case we now define the analytic torsion $T_X(\tau) \in \mathbb{R}^+$ associated to the the flat bundle E_τ, equipped with the admissible metric, by

$$T_X(\tau) := \prod_{p=1}^d \det \Delta_p(\tau)^{(-1)^{p+1}p/2}. \tag{7.14}$$

Let

$$K(t,\tau) := \sum_{p=1}^d (-1)^p p \operatorname{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}). \tag{7.15}$$

If $\tau \ncong \tau_\theta$, then $K(t,\tau) = O(e^{-ct})$ as $t \to \infty$, and the analytic torsion is given by

$$\log T_X(\tau) = \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}K(t,\tau)\,dt\right), \tag{7.16}$$

where the right hand side is defined near $s = 0$ by analytic continuation.

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